VIOLA'S ZONE-LAW

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ABSTRACT

Viola's zone-law, that has been quoted without comments in textbooks, is not a general law. It is only correct if the crystallographic axes and the unit face have been chosen according to the orthodox crystallographic rules with respect to the symmetry elements; then however it holds true in more cases than had been foreseen by Viola.

In his textbook of Mineralogy (1) Niggli states the following thesis: "The face obtained by simple addition of the indices of two equivalent faces—for instance, $h_3=(h_1+h_2)$; $k_3=(k_1+k_2)$; $l_3=(l_1+l_2)$—makes equal angles with these faces. It truncates the edge symmetrically."* In his edition of Klockmann's textbook of Mineralogy, P. Ramdohr makes a slight addition to the thesis and states: "Since a symmetrical truncation occurs *only* where the faces are equivalent, this problem finds its application chiefly on forms with many faces, thus especially in the cubic system" (2).

The same or nearly the same statements are in older textbooks and as early as 1905 C. Viola tried to give a *general* proof of this thesis (3). His proof is however far from correct, mainly as a consequence of a mistake in the application of the transformation of indices, but even besides these errors there are other inaccuracies in his proof. To demonstrate this we will first reproduce Viola's reasoning.

Let two equivalent faces h and k be given. The symbols of these faces are $h=(h_1h_2h_3)$ and $k=(k_1k_2k_3)$. From these the following faces are derived: $m=(h_1+k_1,\ h_2+k_2,\ h_3+k_3)$ and $n=(h_1-k_1,\ h_2-k_2,\ h_3-k_3)$. The statement is, that these faces m and n

(a) bisect the angles included by the faces h and k.

To prove this statement Viola considers two cases, namely:

(1) the faces h and k are equivalent by reflection in a plane of symmetry;

(2) the faces h and k are equivalent by rotation around an axis of symmetry.

1st Case. The faces h and k are symmetrical about a plane of symmetry.

We transform the coordinates in such a manner that the symmetry plane has the symbol (010) and that (001) signifies a plane that is normal to this "new" (010). If

(b) then the "new" indices of the face h be $x_1x_2x_3$, those of the face k must be $x_1\bar{x}_2x_3$. Consequently the "new" indices of the derived faces m and n are respectively:

$$m \cdots (x_1+x_1,\ x_2+\bar{x}_2,\ x_3+x_3) \text{ or } (2x_1,\ 0,\ 2x_3)$$
$$n \cdots (x_1-x_1,\ x_2-\bar{x}_2,\ x_3-x_3) \text{ or } (0,\ 2x_2,\ 0) \text{ or } (010);$$

i.e., the face n coincides with the symmetry plane and the face m is normal to this plane and belongs to the zone $[h, k]$. Since (010) bisects the angle between h and k the face m bisects the complementary angle.

2nd Case. The faces h and k are symmetrical about an axis of symmetry.

We let the direction [001] coincide with the axis of symmetry and we take the "new" [100] and [010] perpendicular to that axis. Let the "new" indices of the face

* Quotations translated in English.

(c) h be $x_1x_2x_3$, then those of the face k are $y_1y_2x_3$. Consequently the symbols of the de-
(d) rived faces m and n are: $m \cdots (x_1+y_1,\ x_2+y_2,\ 2x_3)$ and $n \cdots (x_1-y_1,\ x_2-y_2,\ 0)$. Since the third index of n is zero, the face n belongs to the zone [001]. Taking our next step, we choose this face n as (100). This means that now $x_2=y_2$ and therefore the "new" symbols of our four faces are:

$$h \cdots (x_1x_2x_3), \qquad k \cdots (y_1x_2x_3),$$
$$m \cdots (x_1+y_1,\ 2x_2,\ 2x_3), \quad n \cdots (x_1-y_1,\ 0,\ 0).$$

We now perform a third transformation, taking (010) normal to (100). Since this
(e) transformation lies in the zone [001] the indices x_1 and y_1 must be equal in value and this in such a manner that $x_1=-y_1$, for if $x_1=y_1$ all indices of the face n would become zero and this is impossible. The "newest" symbols of our four faces are therefore:

$$h \cdots (x_1x_2x_3),\ k \cdots (\bar{x}_1x_2x_3),\ m \cdots (0,\ 2x_2,\ 2x_3) \text{ and } n \cdots (2x_1,\ 0,\ 0).$$

The face m lies in the zone [100] and the face n is the face (001) itself; thus they are perpendicular and consequently m and n bisect the angles included by the faces h and k. It is seen that the faces h and k are harmonically separated by the derived faces m and n.

Such was the proof given by Viola; our comments follow.

Comments on the Proof of Viola

The statement represented by (**a**) is not generally true, as can be seen in the following examples (Figs. 1 and 2). Figure 1 is a part of the

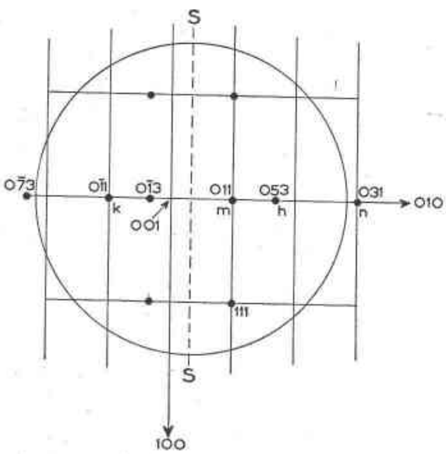

Fig. 1

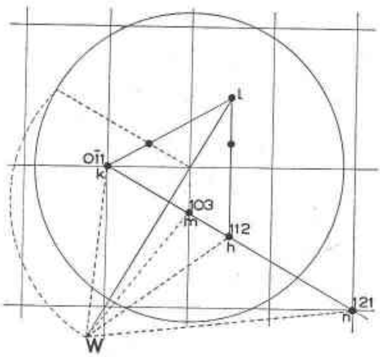

Fig. 2

gnomonic projection of a certain crystal; a plane normal to the vertical crystallographic axis has been selected as the plane of projection and the center of projection is situated upon the vertical line through the center of the circle at a distance equal to the radius of this "gnomon-circle." The pole (001) is to the left of the circle's center and the poles ($0kl$) are seen aligned as usual on the line that points to the pole (010). The poles ($0\bar{1}1$), (001), (011), (021), (031), etc., divide this line in the usual manner in equal parts and the distance of the pole (053) to (001) is 5/3 times the length of those equal parts. With our special crystal this line happens to pass through the foot of the gnomon (=center of the circle) and moreover the distances of the poles ($0\bar{1}1$) and (053) to the center of the circle are by chance alike. Hence the faces $h=(053)$ and $k=(0\bar{1}1)$ are symmetrical about the plane that is normal to the plane of projection along the line SS. These faces being therefore equivalent in the sense of Viola's thesis, the statement of which is that the faces $m=(044)$ or (011) and $n=(062)$ or (031) would bisect the angles included by the faces h and k. A glance at the projection suffices to show that this is not true.

It is easily seen that the peculiarity of this gnomogram is, that notwithstanding the fact that the crystal has a plane of symmetry, the plane (001) is not normal to the symmetry plane. That means that, in contravention of the orthodox rules, the crystallographic b-axis is not normal to the symmetry plane, but that some other edge of the crystal has been chosen for the b-axis.

Figure 2 is a part of the gnomonic projection of another crystal. The poles ($0\bar{1}1$), (103), (112), (121) have been inserted by applying the usual rules to the primitive parallelograms, which here happen to be rectangles in which the long side is $\sqrt{3}$ times the other side. In consequence of this peculiarity the poles ($0\bar{1}1$), (112), ($\bar{1}12$) are the corners of an equilateral triangle, that has its central point in the center of the circle. Hence the faces $h=(112)$ and $k=(0\bar{1}1)$ are symmetrical about a threefold axis of symmetry (in the circle's center normal to the plane of projection). Having constructed the angle point W to the zone [h, k], one sees immediately that the faces $m=(103)$ and $n=(121)$ do not bisect the angles included by the faces h and k. In this case too the crystallographic axes have been chosen in an unorthodox manner, resulting in the peculiarity that, in spite of the axis of threefold symmetry, the three faces of the form have different figures in their symbols.

The conclusion of Viola indicated by (**b**) is not generally correct as is easily seen from Fig. 1. In this gnomogram h and k are symmetrical about the plane of symmetry SS; (010) coincides with the plane of symmetry and (100) is normal to it so that Viola's conditions are all satisfied. Yet the indices of h are 053 and those of k not $0\bar{5}3$ but $0\bar{1}1$.

The conclusion (**c**) does not hold as is seen in Fig. 2: [001] coincides with the axis of symmetry; [100] and [010] are perpendicular to [001]; the faces h and k are related about the symmetry axis and yet the indices of h are 112 and those of k are $0\bar{1}1$ instead of $11\bar{2}$.

The statement (**d**) by Viola makes an implicit assumption about the transformation. In the following paragraph it will be demonstrated that his assumption is incorrect.

The passage indicated by (**e**) is rather obscure.

Viola takes for granted that on transforming indices, the face having for its "new" indices the sum (or difference) of the "new" indices of two faces h and k, is the same as the face having for its "old" indices the sum (or difference) of the "old" indices of the faces h and k. This however is not necessarily true.

Let (pqr) be the original symbol of a face and $(p'q'r')$ its new symbol, then the relations between the two take the following form: $p':q':r' = f_1:f_2:f_3$ where f_1, f_2, f_3 are linear functions of p, q and r. This does not mean that $p'=f_1$, $q'=f_2$, $r'=f_3$ because the orthodox crystallographic rules demand the indices of a face be reduced to coprime integers.

Taking now the faces $\pi_1=(p_1q_1r_1)$ and $\pi_2=(p_2q_2r_2)$, we want to check whether the face $\pi_3=(p_1+p_2, q_1+q_2, r_1+r_2)$ is the same as the face π_4, the symbol of which is found by adding the new indices p_1', q_1', r_1' and p_2', q_2', r_2' of π_1 and π_2. The relations are $p_1'=a_1f_1$, $q_1'=a_1f_2$, $r_1'=a_1f_3$;

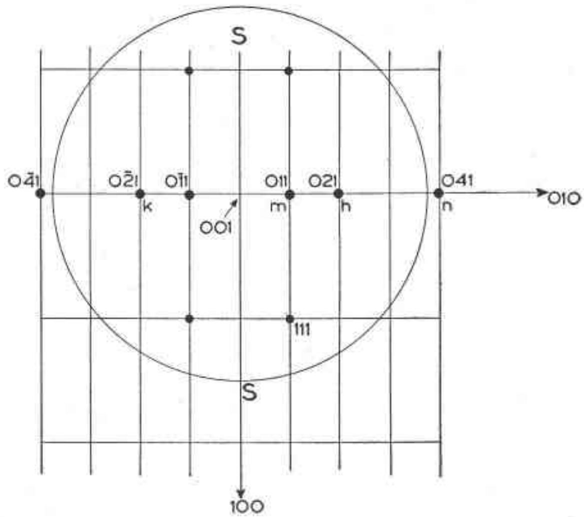

FIG. 3

$p_2'=a_2\phi_1$, $q_2'=a_2\phi_2$, $r_2'=a_2\phi_3$. The face π_3 whose "old" unsimplified indices are $p_1+p_2, q_1+q_2, r_1+r_2$ receives for new indices $a_3(f_1+\phi_1), a_3(f_2+\phi_2), a_3(f_3+\phi_3)$. On the other hand the indices of the face π_4 are $a_4(a_1f_1+a_2\phi_1)$, $a_4(a_1f_2+a_2\phi_2)$, $a_4(a_1f_3+a_2\phi_3)$. Hence π_3 and π_4 are only the same face if $(f_1+\phi_1):(f_2+\phi_2):(f_3+\phi_3)=(a_1f_1+a_2\phi_1):(a_1f_2+a_2\phi_2):(a_1f_3+a_2\phi_3)$. In general these relations will not necessarily hold and therefore Viola's implicit assumption was not warranted. There are however special cases (if for instance $a_1=a_2$) in which the above relation will be satisfied and hence $\pi_3=\pi_4$.

An example of a general case can be studied by comparing Fig. 3 with Fig. 1. In these gnomograms the positions of the faces are identical but the primitive parallelograms are different because of a difference in the choice of the crystallographic axes. Calling the crystallographic description of Fig. 3 the "old" and that of Fig. 1 the "new" setting, one finds that the "old" symbols of the "new" axes are [100], [021] and [001]. The unit face being the same in both cases, the relation between the new and the old indices of a face is

$$p':q':r' = p:\frac{2q+r}{3}:r.$$

Taking now (Fig. 3) the faces $(021)=\pi_1$ and $(0\bar{2}1)=\pi_2$, one finds $\pi_3 = (002) = (001)$. Further

$$p_1' = 0, \quad q_1' = 3\,\frac{2\times 2+1}{3}, \quad r_1' = 3\times 1 \text{ (thus } a_1 = 3)$$

$$p_2' = 0, \quad q_2' = \frac{2\times \bar{2}+1}{3}, \quad r_2' = 1 \text{ (thus } a_2 = 1).$$

Therefore

$$\pi_4 = (0+0,\ 5+\bar{1},\ 3+1) = (011) \text{ (Fig. 1) or } (011) \text{ (Fig. 3)}.$$

Hence π_3 and π_4, in this case, are different faces.

The weak point in Viola's argument lies in the fact that he tries to give a *general* demonstration for a thesis that has only a *restricted* validity. Viola speaks about indices without mentioning the crystallographic axes and the unit face that gives those indices their very meaning. Obviously he thinks he can prove his thesis if the axes and the unit face are, respectively, three arbitrary non coplanar crystal edges and an arbitrary suitable crystal face. We saw however that under these circumstances the thesis has no general validity. If, on the contrary, the axes and the unit face have been chosen under guidance of the crystal symmetry according to the orthodox crystallographic rules, it is a different matter. Then the thesis is true and this might be the cause of the unusual fact that for forty years no objections have been raised against Viola's argument.

On the assumption that the axes and the unit face have been chosen according to the usual crystallographic rules, we will now proceed to consider the various crystal systems.

Cubic System. Take two faces $(h_1k_1l_1)$ and $(h_2k_2l_2)$ of a hexoctahedron $\{hkl\}$; from the center drop perpendiculars n_1 and n_2 and observe that these have the same length d.* Resolving n_1 and n_2 in the directions of

$$* d = \frac{a}{\sqrt{h_1^2+k_1^2+l_1^2}} = \frac{a}{\sqrt{h_2^2+k_2^2+l_2^2}}$$

the crystallographic axes we find the following components:

$$d^2 \times \frac{h_1}{a};\ d^2 \times \frac{h_2}{a};\ d^2 \times \frac{k_1}{a};\ d^2 \times \frac{k_2}{a};\ d^2 \times \frac{l_1}{a};\ d^2 \times \frac{l_2}{a}.$$

Therefore the components of the resultant of n_1 and n_2 are:

$$d^2 \times \frac{h_1 + h_2}{a};\quad d^2 \times \frac{k_1 + k_2}{a};\quad d^2 \times \frac{l_1 + l_2}{a}.$$

Hence the resultant is perpendicular to the face $(h_1+h_2,\ k_1+k_2,\ l_1+l_2)$. But the resultant of two equal vectors bisects the angle included by those vectors and therefore the face $(h_1+h_2,\ k_1+k_2,\ l_1+l_2)$ is normal to the bisectrix of the angle between the faces $(h_1k_1l_1)$ and $(h_2k_2l_2)$, or in other words: the face $(h_1+h_2,\ k_1+k_2,\ l_1+l_2)$ bisects one of the angles inlcuded by $(h_1k_1l_1)$ and $(h_2k_2l_2)$.

From this proof it will be clear that in the cubic system *the thesis is correct in all cases of two faces having the same distance d to the center, i.e.* not only for two faces of the same crystal form but also for two faces belonging to different forms, such as {322} and {410}, or {510} and {431}, or {552} and {432}, etc. (cp. *Int. Tables Det. Cryst. Str.*, Vol. **II**, Chap. IX, p. 6). So, on chosing for $(h_1k_1l_1)$ any of the 48+48 faces {322} +{410} and for $(h_2k_2l_2)$ any other of those 96 faces, one finds the faces $(h_1+h_2,\ k_1+k_2,\ l_1+l_2)$ and $(h_1-h_2,\ k_1-k_2,\ l_1-l_2)$ to bisect the angles included by the former faces.

The Tetragonal, Hexagonal, Orthorhombic and Monoclinic Systems. In any gnomogram, the projection plane of which is normal to the crystallographic c-axis, the following theses hold true:

(a) The symbol of the middle point of the line joining the poles (h_1k_1l) and (h_2k_2l) is $(h_1+h_2,\ k_1+k_2,\ 2l)$, while the infinitely far point of the zone line $[(h_1k_1l),\ (h_2k_2l)]$ has the symbol $(h_1-h_2,\ k_1-k_2,\ 0)$.

(b) The angle-point W of a zone line is on the perpendicular drawn from the center C of the gnomon-circle on that zone line. In case this perpendicular passes through the middle point of the line joining (h_1k_1l) and (h_2k_2l), the faces $(h_1+h_2,\ k_1+k_2,\ 2l)$ and $(h_1-h_2,\ k_1-k_2,\ 0)$ bisect the angles included by the faces (h_1k_1l) and (h_2k_2l).

(c) Given a pole p_1, the pole p_2 satisfying the condition that the "derived" faces, whose symbols are deduced from those of p_1 and p_2 by addition and subtraction, bisect the angles included by p_1 and p_2, is found as follows:

(1) draw with the point C as center a circle passing through the pole p_1;

(2) any pole in the circumference of this circle having its third index equal to the third index of p_1 meets the qualifications of p_2.

By way of example, Fig. 4 is a gnomonic projection of a ditetragonal bipyramid $\{hkl\}$, with the constructions of the angle-points W_1 and W_2 of the zone lines $[(hkl), (\bar{k}hl)]$ and $[(h\bar{k}l), (k\bar{h}l)]$. It will be clear at once that the lines which join the poles (hkl) and $(\bar{k}hl)$ to the center C form with

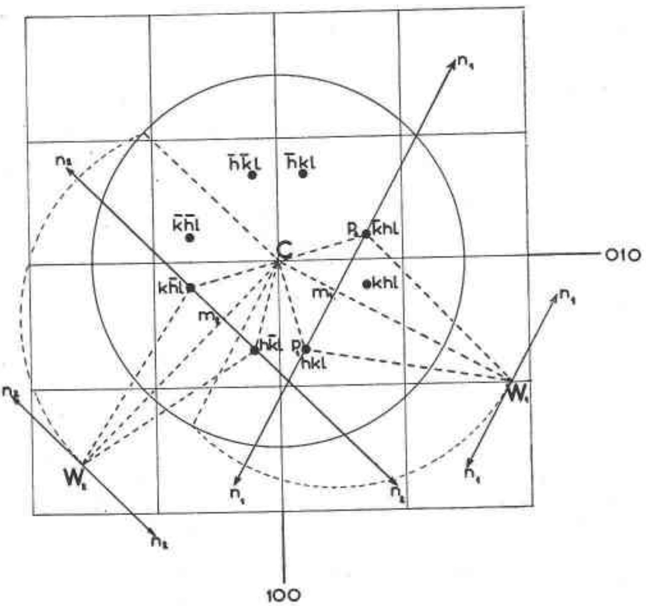

FIG. 4

the zone line an isosceles triangle and that therefore $\angle p_1W_1m_1 = \angle p_2W_1m_1$. Hence the faces m_1 and n_1 bisect the angles included by p_1 and p_2.

Taking into acount Goldschmidt's gnomonic theorem one finds for the indices of m_1 and n_1 $\left(\frac{h-k}{2l}, \frac{k+h}{2l}, 1\right)$ or $(h-k, k+h, 2l)$ and $(h+k, k-h, 0)$. Taking the side of the "primitive square" as unit of length, the distance of a pole (hkl) to the center C in a tetragonal gnomogram is $\sqrt{\left(\frac{h}{l}\right)^2 + \left(\frac{k}{l}\right)^2}$. Hence the poles (h_1k_1l) and (h_2k_2l) are on the circumference of a circle whose center is in C, if $h_1^2+k_1^2=h_2^2+k_2^2$. Therefore Viola's thesis holds true in the tetragonal system for any two faces (h_1k_1l) and (h_2k_2l) satisfying the so-called "quadratic form" $h_1^2+k_1^2 = h_2^2+k_2^2$ *i.e.*, not only for any two faces of a crystal form but also for the faces (671) and (291), for instance. To take into account faces whose third index has the negative sign it is a good plan to replace a face $(pq\bar{l})$ by the parallel face $(\bar{p}\bar{q}l)$, which has its pole in the gnomogram.

In a hexagonal gnomogram the distance of a pole (hkl) to the center C is $2\sqrt{\left(\frac{h}{l}\right)^2+\left(\frac{k}{l}\right)^2+\frac{h}{l}\,\frac{k}{l}}\cdot$ Hence the poles (h_1k_1l) and (h_2k_2l) are on the circumference of a circle whose center is in C, if $h_1^2+k_1^2+h_1k_1 = h_2^2+k_2^2+h_2k_2$. Therefore Viola's thesis holds true in the hexagonal

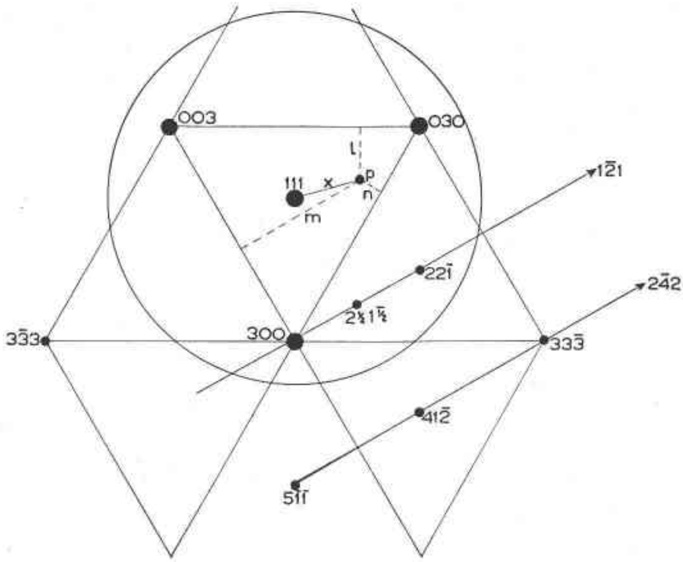

FIG. 5

system for any two faces (h_1k_1l) and (h_2k_2l) satisfying the quadratic form $h_1^2+k_1^2+h_1k_1=h_2^2+k_2^2+h_2k_2$, *i.e.*, not only for any two faces of a crystal form, but also for the faces (652) and (912), for instance.

The orthorhombic system offers no difficulties, nor does the monoclinic system, provided the plane of projection in the latter case is chosen normal to the crystallographic *b*-axis. In both systems Viola's thesis holds true for any two faces of a crystal form.

Trigonal crystals that are described in Miller's system must be treated separately because, in their gnomograms, the plane of projection is not normal to any crystallographic axis. Consequently with these crystals the indices are read off in quite a different manner, namely by drawing through the pole p the perpendicular distances l, m, n to the sides of the "base-triangle" (Fig. 5) and seeking three numbers in the ratios $l:m:n$, whose sum is 3.

The distance of a pole p to the center of the gnomon circle is

$$\tfrac{2}{3}\sqrt{(l^2 + m^2 + n^2) - (lm + ln + mn)}.$$

Therefore the indices $(p_1q_1r_1)$ and $(p_2q_2r_2)$ of two poles that are at the same distance from the center of the gnomon circle satisfy the condition:

$$p_1^2+q_1^2+r_1^2-(p_1q_1+p_1r_1+q_1r_1)=p_2^2+q_2^2+r_2^2-(p_2q_2+p_2r_2+q_2r_2).$$

Besides any two faces of the same crystal form, there are many pairs of faces belonging to different forms that satisfy this condition; for instance, (300) and $(22\bar{1})$ or $(5\bar{1}\bar{1})$ and $(33\bar{3})$. More examples can easily be found with the help of *Int. Tab. Det. Cryst. Struct.*, Vol. **II**, Chap. IX, p. 4. In all those cases Viola's thesis holds true, *i.e.* the "derived" faces bisect the angles included by the "given" faces. The sum of the indices of the face found by adding the indices of the given faces will be 6. Before inserting this symbol in the gnomogram, the indices may therefore be divided by 2, in order to maintain the rule that the sum of the indices of a face be always 3. The sum of the indices of the face found by subtracting the indices of the given pair of faces is zero. This indicates that the face is parallel to the zone [111] and that its pole is infinitely far away. Examples are: $(5\bar{1}\bar{1})$ and $(33\bar{3})\rightarrow(41\bar{2})$ and $(2\bar{4}2)$, or (300) and $(22\bar{1})\rightarrow(2\tfrac{1}{2}, 1, \bar{\tfrac{1}{2}})$ and $(1\bar{2}1)$.

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