CHARTING FIVE, SIX, AND SEVEN VARIABLES ON HYPERTETRAHEDRAL FACES

JOHN B. MERTIE, JR.[1]

CONTENTS

ABSTRACT

Three variables may be charted in a triangle by the use of trilinear coordinates, and four variables in a tetrahedron by means of quadriplanar coordinates. Negative trilinear and quadriplanar coordinates may also be used to advantage. These unique properties of the triangle and tetrahedron suggest that similar properties exist for hypertetrahedra of n dimensions.

By mathematical generalization, it is possible to predict the number of vertices, edges, triangular faces, tetrahedra, and hypertetrahedra that bound an n-dimensional hypertetrahedron. A tabulation of these boundaries, up to the ninth dimension, is given. The number of vertices in each hypertetrahedron corresponds to the number of variables that may be charted within it. Hypertetrahedra of 4, 5, and 6 dimensions, having 5, 6 and 7 vertices, are bounded respectively by 10, 20, and 35 triangular faces.

Four variables may be charted in a tetrahedron by constructing geometrically or by deducing analytically the resulting surface, which may then be shown either in perspective or by projection as a topographic map. Another method is to develop the tetrahedron onto a plane, and merely to plot the triads 123, 124, 134, and 234, each recomputed to 100 per cent. The first method is inapplicable to hypertetrahedra, and the second method may not be exactly applied, as hypertetrahedra can not be developed. The triangles bounding the hypertetrahedra, however, may be arranged empirically, so as to constitute a compound system of trilinear coordinates for charting the triads 1, $2 \cdots (v-1)$, $3 \cdots v$, 2, $3 \cdots (v-1)$, $4 \cdots v$, 3, $4 \cdots (v-1)$, $5 \cdots v$, etc., where v means both vertices and variables.

Coordinate systems of this kind have been prepared for charting 5, 6, and 7 variables. Some choice exists in the arrangement of the triangles, but the factor of compactness practically eliminates all but one arrangement. Such coordinates may be given algebraic meanings within individual triangles, but not between them. The composite charts, however, afford geometrical pictures which, if conventionalized, may be as effective as a true system of analytical coordinates.

Four analyses of platinum metals are used to illustrate the charting of 6 variables on the 20 triangular faces of a hypertetrahedron of 5 dimensions. Variable scales are required for best delineation of the resulting curves, and methods are given for producing such changes in scale.

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Introduction

Physical measurements commonly involve three or more components or variables, and the relations between them are most readily grasped if they can be shown in graphic form. Numerous systems of 3-dimensional coordinates have been devised for this purpose, but 3-dimensional rectangular cartesian coordinates are generally used. Any equation in three variables may thus be charted, but an equation in four variables presents difficulties. A 3-dimensional surface may be projected orthogonally onto a plane, thus reducing by one its number of dimensions, and producing a topographic map. Similarly, but by the use of analytical methods, an equation in four variables, representing a 4-dimensional continuum, may be projected into three dimensions; but the resulting representation will comprise a series of 3-dimensional surfaces, as numerous as the contours on a topographic map. Graphs of this kind are impracticable for four variables, and are almost impossible for five or more variables.

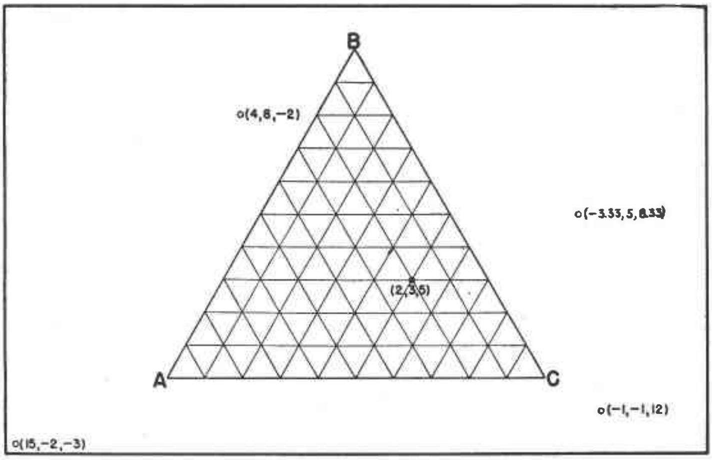
All systems of coordinates for charting more than three variables are impaired by shortcomings of this, or some other kind. Partial success has been achieved by combining components, by making multiple curves or diagrams, by the use of nomograms, and by other devices, but restrictions of some sort are invariably required. One of the limitations that may be tolerated, particularly in charting chemical analyses, is that the sum of the variables shall equal unity, or 100 per cent. With this limitation, three variables may be charted within a triangle, and four within a tetrahedron. This unique property, possessed by the triangle and tetrahedron, is also possessed by hypertetrahedra of n dimensions, so that any number of components whose sum is unity may theoretically be charted. But it is impossible to depict graphically such n-dimensional continua, so that further restrictions and compromises must be made. This paper is an exposition of one method for utilizing the concept of hypertetrahedral charting.

Trilinear and Quadriplanar Coordinates

Trilinear coordinates are so commonly used in the graphic representation of 3 variables that no description of this usage seems necessary. Little application is made, however, of the analytical geometry of trilinear coordinates, whereby trilinear equations may be written to show relationships between curves that are experimentally derived and graphically presented. American textbooks on elementary analytical geometry are particularly reticent on this subject. Two elementary statements and one complete treatise on trilinear coordinates, all of British origin, are cited herewith:

SMITH, CHARLES, An elementary treatise on conic sections by the methods of coordinate geometry. Macmillan & Co., Ltd., London, pp. 341–389 (1919).
LONEY, SIDNEY L., The elements of coordinate geometry: part II, trilinear coordinates. Macmillan & Co., Ltd., London, 228 pp. (1923).
WHITWORTH, WILLIAM A., Trilinear coordinates and other methods of modern analytical geometry of two dimensions. Deighton, Bell & Co., London, 506 pp. (1866).

Trilinear coordinates are built about any 3 non-parallel non-concurrent lines in a plane, which intersect to form a triangle of reference, commonly called a trigon. This triangle may have any shape, but both



POSITIVE AND NEGATIVE TRILINEAR COORDINATES

FIG. 1

graphics and analysis are simplified if an equilateral triangle is used. Such a trigon is illustrated by Fig. 1. The three coordinates of a point, written as (α, β, γ), are measured on normals to the edges, drawn in the direction of the opposite vertices for positive values, and in the reverse direction for negative values. One or two coordinates may be negative, but not three.

If the sides of a trigon are represented as $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$, and its area as Δ, it is readily shown that

$$\frac{\mathfrak{a}\alpha + \mathfrak{b}\beta + \mathfrak{c}\gamma}{2\Delta} = 1. \tag{1}$$

Any term of any trilinear equation may therefore be multiplied one or more times by the left side of equation (1), without changing the value of the equation. Hence all algebraic trilinear equations are, or can be rendered, homogeneous. Thus the general equations for a straight line and for a conic section are respectively as follows:

$$l\alpha + m\beta + n\gamma = 0$$

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0.$$

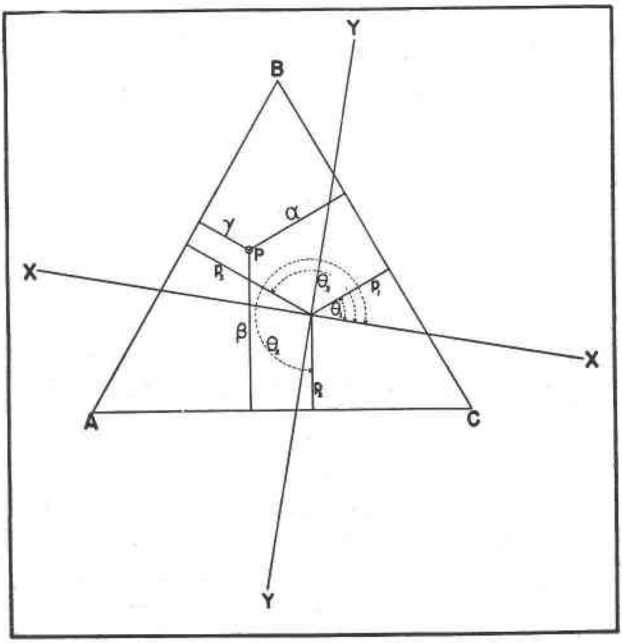
Trilinear coordinates are readily transformed to rectangular cartesian coordinates by means of the following formulae:

$$\alpha = x \cos \theta_1 + y \sin \theta_1 - p_1 \quad (2)$$

$$\beta = x \cos \theta_2 + y \sin \theta_2 - p_2 \quad (3)$$

$$\gamma = x \cos \theta_3 + y \sin \theta_3 - p_3 \quad (4)$$

where p_1, p_2, and p_3 are the trilinear coordinates of the origin of cartesian coordinates, and θ_1, θ_2, and θ_3 are the angles which normals from a point



TRANSFORMATION OF COORDINATES

FIG. 2

to the sides of the trigon make with the X axis. These relationships are shown in Fig. 2. For further information on trilinear coordinates, the reader is referred to the publications cited above.

Quadriplanar coordinates bear the same relation to a tetrahedron as trilinear coordinates do to a triangle. Commonly a regular tetrahedron of reference is used, having equilateral triangular faces. Positive coordinates are measured on a normal from each face in the direction of the opposite apex; negative coordinates are measured in the reverse direction. One, two, or three negative coordinates may exist, but not four. All algebraic quadriplanar equations are, or may be rendered, homogeneous by the same method heretofore shown for trilinear coordinates; and the transformation to 3-dimensional rectangular cartesian coordinates is made by the use of four formulae analogous to (2), (3), and (4).

Application of Negative Coordinates

The point (2, 3, 5) is one of those charted in Fig. 1. Its coordinates may be considered to represent the composition of a rock composed of 2 parts quartz, 3 parts feldspar, and 5 parts mafic minerals. Suppose another rock exists whose composition is 2 parts nepheline, 3 parts feldspar, and 5 parts mafic minerals. As quartz and nepheline are incompatible, the mode of this alkaline rock might be given as $(-2, 3, 5)$. But the sum of these numbers is 6 instead of 10, wherefore each must be multiplied by $\frac{10}{6}$ to produce the true coordinates $(-3.33, 5, 8.33)$. This point may then be charted, as shown in Fig. 1, to represent the composition of the alkaline rock. If in some comagmatic region, other igneous rocks exist whose modes lie between or beyond the points (2, 3, 5) and $(-3.33, 5, 8.33)$, they may also be charted, and their loci may be joined by a fitted curve whose trilinear equation can be written with reference to the trigon *ABC*. This would not be possible if the alkaline rocks had been charted with reference to a contiguous trigon, say *ACD*.

The expansion shown above, to obtain coordinates whose algebraic sum equals 10, is more generally accomplished by multiplication by $L/[S]$, where L is the number of divisions into which each side of the trigon is divided, and $[S]$ is the absolute value of the algebraic sum of the original ratios of the mode. Attention is called to the fact that α, β, and γ become points at infinity if $S=0$. The original ratios may be recovered from the expanded coordinates by means of the equation

$$r = \frac{L\alpha}{[\alpha] + [\beta] + [\gamma]}$$

where r refers to one of the three original ratios of the mode, and α refers to its coordinate. Similar formulae, of course, are used for s and t, the other two ratios of the mode, and for β and γ, their derived coordinates. The use of negative coordinates thus permits the analytic charting of four variables in a plane, if two of these variables are incompatible; and of five variables in a plane, if two pairs of variables are incompatible.

Negative quadriplanar coordinates may also be used to advantage, as shown in the following example. The six platinum metals, when analyzed, are first treated in hot aqua regia, but an insoluble residue remains that must be fused with a flux. The soluble and insoluble fractions are separately analyzed, and are afterwards combined in proper proportions to give the complete analysis. The soluble fraction contains platinum, iridium, rhodium, and palladium; the insoluble fraction contains platinum, iridium, rhodium, osmium, and ruthenium. Palladium is thus absent from the insoluble fraction, and osmium and ruthenium are absent from the soluble fraction. The analysis of the soluble fraction may be charted in quadriplanar coordinates as a point within a tetrahedron of reference whose vertices are Pt, Ir, Rh, and Pd. By combining the hexagonal elements osmium and ruthenium, and considering them as incompatible with palladium, the component Os-Ru may be plotted as negative in relation to the vertex Pd. Thus the analysis of the insoluble fraction may be represented as a point outside the same tetrahedron of reference. A series of such analyses can therefore be represented as two surfaces, one inside and the other outside the tetrahedron; and thereafter both surfaces may be shown as topographic maps. Similarly six variables may be charted in quadriplanar coordinates, if two pairs of variables are incompatible; and seven variables may be charted if three pairs are incompatible.

Hypertetrahedral Charting

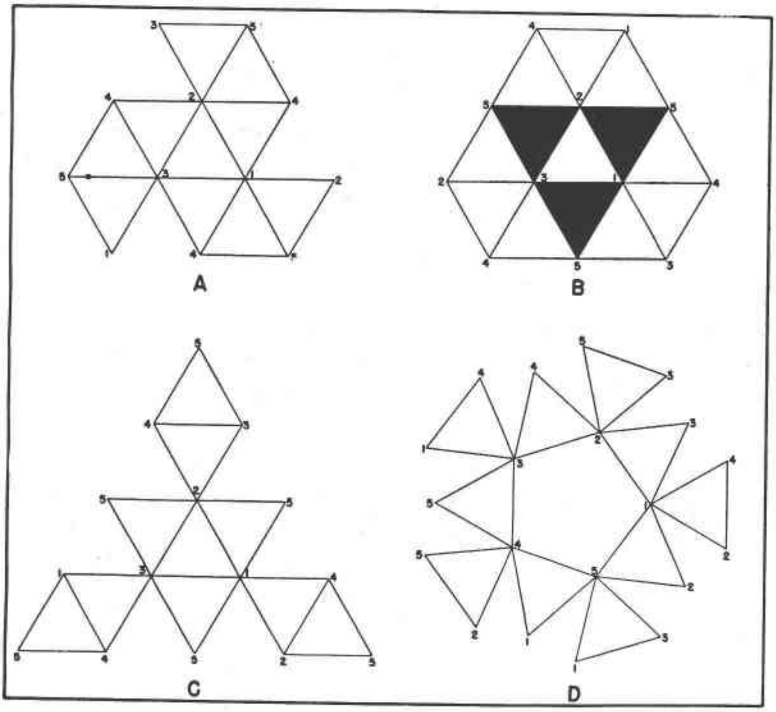
From the properties of the triangle and the tetrahedron, it follows by mathematical induction that n variables may be charted in hypertetrahedra of n-1 dimensions. The boundaries of such hypertetrahedra, up to the ninth dimension, are shown in the following tabulation:

Dimensions of Hypertetrahedra

Boundaries	Fourth	Fifth	Sixth	Seventh	Eighth	Ninth
Vertices	5	6	7	8	9	10
Edges	10	15	21	28	36	45
Triangles	10	20	35	56	84	120
Tetrahedra	5	15	35	70	126	210
H_4	1	6	21	56	126	252
H_5	0	1	7	28	84	210
H_6	0	0	1	8	36	120
H_7	0	0	0	1	9	45
H_8	0	0	0	0	1	10
H_9	0	0	0	0	0	1

In this tabulation, H_4, H_5, etc. refer to hypertetrahedra of the fourth, fifth, and higher dimensions.

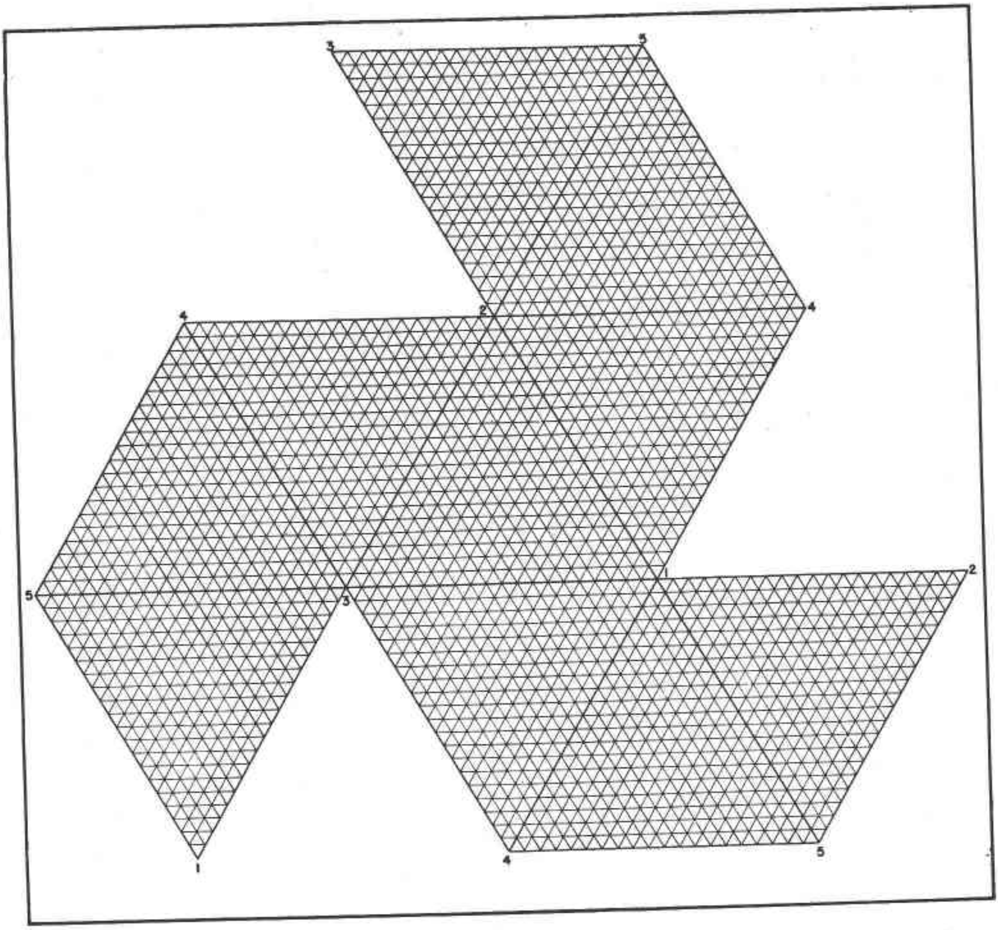
The charting of continua, and their fitting to experimental data, are possible and feasible by analytical methods, but unfortunately no practical method exists for a graphic presentation of the results. Compromises must therefore be sought. Consider the problem of charting 5 variables. At first sight it might seem that the best method would be to utilize quadriplanar coordinates, charting the data in the tetrahedra that bound



FOUR ARRANGEMENTS OF TEN TRIGONS

FIG. 3

a hypertetrahedron of 4 dimensions. But there are 5 such tetrahedra, so that it would be necessary to chart a surface in each of the tetrahedra 1234, 1235, 1245, 1345, and 2345, only two of which could have common bases. This would be a laborious procedure that few would attempt. An alternative would be to develop the 5 tetrahedra, preserving each developed tetrahedron as a unit, and showing the triad relationships on 20 triangular faces. But this is wasteful of space, because 10 faces bound a hypertetrahedron of 4 dimensions, and therefore only 10 triangles need to be shown to convey the same amount of information.



TRILINEAR COORDINATES FOR FIVE VARIABLES

Fig. 4

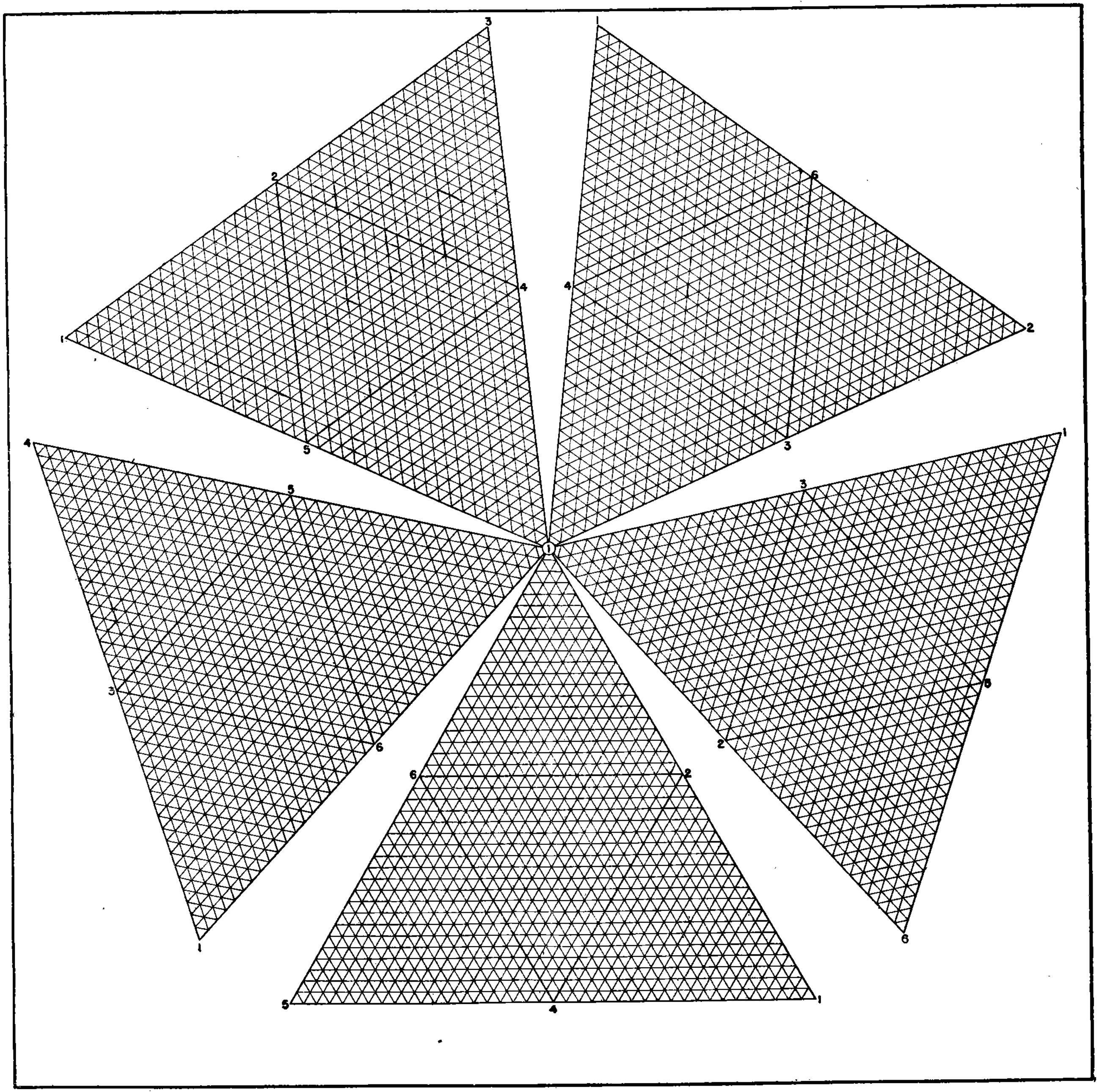
The simplest arrangement for the 10 triangles 123, 124, 125, 134, 135, 145, 234, 235, 245, and 345 is in rows and columns, as in a rectangular array, but such an assembly would be uneconomical of space, and would show no relationships between adjoining triads. When assembled as a single diagram, however, some choice still exists in the arrangement of trigons, but conservation of space largely eliminates such choice. Figure 3, for example, shows 4 ways in which the 10 trigons mentioned above may be arranged. Arrangements *C* and *D* require more space than *A* or *B*, and therefore for a diagram of given size, the trigons of *C* and *D* would have to be smaller. Arrangements *A* and *B* have the same size and shape, but *A* preserves one developed tetrahedron, whereas *B* does not. Arrangement *A* is obviously the best one.

The hypertetrahedra of higher dimensions show further difficulty in the utilization of quadriplanar coordinates. It will be noticed, as the number of variables increases, that the number of bounding tetrahedra increases faster than the number of bounding triangles, so that the labor of charting surfaces increases progressively. Thus for 7 variables, the number of tetrahedra equals the number of triangles; but for more than 7 variables, the tetrahedra are more numerous than the triangles. All these considerations have impelled the writer to the use of compound systems of trilinear coordinates, wherein the bounding triangles are shown only once.

The Charts

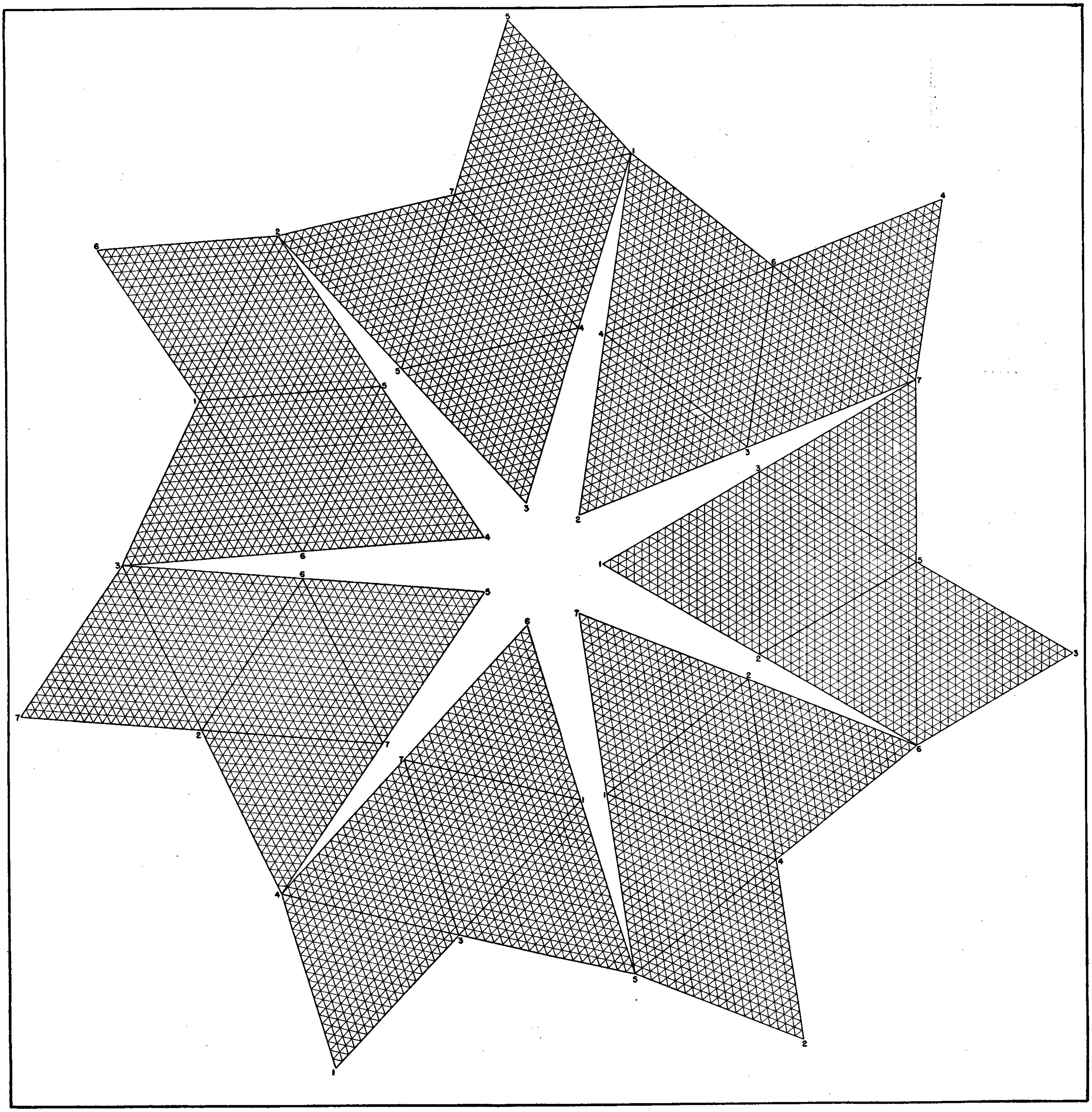
Figures 4, 5, and 6 show the most compact arrangements of the triangular faces that bound hypertetrahedra of 4, 5, and 6 dimensions; and these render possible the charting, respectively, of 5, 6, and 7 variables. A chemical analysis, for example, is computed to total unity, or 100 per cent, after which all possible triads are also recomputed to unity. These values are then plotted in their respective trigons.

One of the difficulties in any system of charting is scale. Certain sets of values may be well represented at one scale, whereas others, plotted at the same scale, will fail to show distinct relationships. This is overcome in cartesian coordinates by a change in scale of the ordinates or abscissae, or both. The same problem necessarily occurs in trilinear coordinates, where several sets of coordinates that should delineate a curve, may be so near to one another that the resulting curve is too small for satisfactory inspection. Amplification of the curve is therefore desirable, but such amplification must not result in the charting of points outside the trigon in which their coordinates belong. The accomplishment of this objective is found to depend upon the minimum value of the largest coordinates among the sets to be charted within a single trigon.



TRILINEAR COORDINATES FOR SIX VARIABLES

Fig. 5



TRILINEAR COORDINATES FOR SEVEN VARIABLES

FIG. 6

The relationship between magnification and coordinates is given by the following formula:

$$Mx - M + 1 = 0 \tag{5}$$

where M is the possible magnification of scale, and x is the minimum value of the largest coordinate in a number of sets. Thus it may happen that the minimum value of the largest coordinate in several sets is not less than .80, from which it follows that the maximum amplification of scale is 5. Attention is directed to the fact that it is immaterial whether the largest coordinates in the sets are α, β, or γ, or a mixture of these. Some of the possible magnifications are shown below:

Possible Magnification (M)	Minimum Value of Maximum Coordinates (x)
$M = 100$	$x \geqq .990$
50	.980
25	.960
15	.933
10	.900
9	.889
8	.875
7	.857
6	.833
5	.800
4	.750
3	.667
2	.500
$1\frac{1}{2}$	.333

The value of a magnified maximum coordinate is obtained from the following formula:

$$V = LMC - L(M - 1) \tag{6}$$

where C is the numerical value of the maximum coordinate, L is the number of divisions of each side of the trigon, M is the magnification, and V is the required value of the maximum coordinate in the magnified scale. Thus, if $C = \alpha = .824$, $L = 20$ (as in Figs. 4, 5, and 6), and $M = 5$ (as indicated from the preceding tabulation), the value of V will be 2.4 divisions of the scale. The value of the magnified minor coordinates is obtained from the equation:

$$v = LMc$$

where c is the numerical value of either of the minor coordinates, and v is the required value of a minor coordinate in the magnified scale.

Magnification of scale is commonly necessary. Each trigon, however, is a separate and distinct unit, so that the methods of trilinear analytical

PLATINUM ANALYSES AND RECOMPUTED TRIADS

Platinum Creek and Fox Gulch

Platinum	70.23	71.90	75.03	73.99	75.12	93.58	91.97	93.72	97.15	99.10	97.31										
Iridium	23.00	23.55	24.57	24.23	24.60							82.68	78.96	83.02	91.79	97.32	92.20				
Osmium	4.45	4.55				5.92	5.82	5.93				15.98	15.26	16.04				68.35	87.53	69.56	
Ruthenium	.37		.40			.50			.52	.53		1.34			1.49	1.58		5.74	7.35		16.10
Rhodium	1.69			1.78			2.21		2.33		2.33		5.78		6.72		6.76	25.91		26.37	72.68
Palladium	.26				.28			.35		.37	.36			.94		1.10	1.04		5.12	4.07	11.22

Squirrel Creek

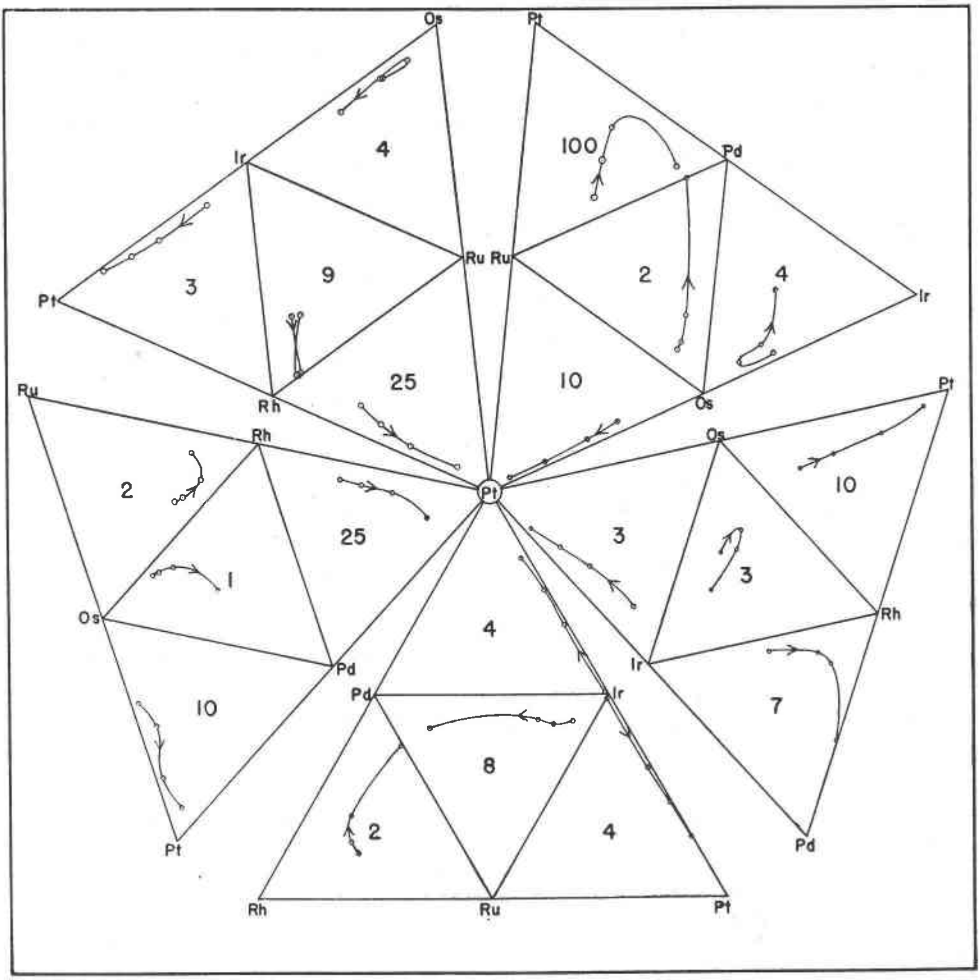
Platinum	78.93	80.69	83.63	82.50	83.61	95.15	93.69	95.13	97.68	99.25	97.66										
Iridium	15.16	15.49	16.06	15.84	16.06							79.03	74.02	78.93	89.00	96.21	88.88				
Osmium	3.73	3.82				4.50	4.43	4.50				19.47	18.24	19.45				66.60	86.21	66.33	
Ruthenium	.29		.31			.35			.36	.36		1.50			1.69	1.82		5.12	6.63		13.16
Rhodium	1.58			1.66			1.88		1.96		1.96		7.74		9.31		9.30	28.28		28.16	72.63
Palladium	.31				.33			.37		.39	.38			1.62		1.97	1.82		7.16	5.51	14.21

Salmon River

Platinum	84.76	86.26	88.11	87.16	87.94	97.25	96.10	97.04	98.39	99.38	98.18										
Iridium	11.27	11.47	11.71	11.59	11.69							82.45	76.63	81.35	89.06	95.50	87.78				
Osmium	2.23	2.27				2.55	2.53	2.55				16.29	15.14	16.07				61.66	80.75	58.66	
Ruthenium	.17		.18			.20			.20	.20		1.26			1.37	1.47		4.79	6.28		9.93
Rhodium	1.21			1.25			1.37		1.41		1.40		8.23		9.57		9.43	33.55		31.92	69.54
Palladium	.36				.37			.41		.42	.42			2.58		3.03	2.79		12.97	9.42	20.53

Clara Creek

Platinum	90.49	91.75	92.40	92.06	91.90	98.89	98.51	98.32	99.25	99.07	98.68										
Iridium	7.29	7.39	7.44	7.41	7.40							87.80	84.16	82.52	91.47	89.53	85.76				
Osmium	.85	.86				.93	.93	.93				10.27	9.84	9.65				55.65	50.00	41.32	
Ruthenium	.16		.16			.18			.18	.17		1.93			2.02	1.97		10.48	9.42		11.71
Rhodium	.52			.53			.56		.57		.57		6.00		6.51		6.10	33.87		25.15	37.84
Palladium	.69				.70			.75		.76	.75			7.83		8.50	8.14		40.58	33.53	50.45



PLATINUM METALS, GOODNEWS BAY DISTRICT, ALASKA

Fig. 7

geometry may be applied within trigons, but not from one trigon to another. Therefore no objection exists to showing different trigons at different scales; and where this is done, the magnification is given as a single large numeral within the trigon. Magnification of scale is impracticable where negative coordinates are utilized.

An example of the charting of 6 variables is shown in Fig. 7. For this purpose, four analyses of platinum metals, taken from a report by the writer[2] are used, each of which is a mean of a number of analyses. The four analyses are shown at the left of the subjoined tabulation, followed to the right by all possible triads recomputed to unity. All but one of the 20 trigons are used at magnified scales. Curves have been drawn through the charted points, to show the modes of variation; and arrows have been placed on the curves to show the relative positions of the four analyses.

Résumé

A method is shown for charting 5, 6, and 7 variables on the triangular faces of hypertetrahedra of 4, 5, and 6 dimensions. The triangles representing these faces are empirically arranged for the maximum conservation of space. The variables are computed into triads, each of which total unity, and are then charted in their respective trigons by the use of trilinear coordinates. Different scales are commonly required in the different trigons, and methods are given for producing such changes in scale. The charting of 6 variables is illustrated by the use of 4 analyses of platinum metals from the Goodnews Bay district, Alaska.

[2] Mertie, J. B., Jr., The Goodnews platinum deposits, Alaska: *U. S. Geol. Survey, Bull.* **918,** 77–79 (1940).